Graph Theory

Colorings, cliques, independent and dominating sets

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A subset $U \subseteq V$ is independent in $G = (V, E)$ if no two vertices in U are neighbors. The **independence number** $\alpha(G)$ is the size of the largest independent set in G. The associated decision problem is NP-complete. It is easy to prove the sum of the independence number and the size of the smallest vertex cover is always equal to the order of the graph.

The **clique number** $\omega(G)$ is the size of the largest clique in G. The associated decision problem is NP-complete (equivalent to independent set).

The **matching number** $\beta(G)$ is the size of the largest matching in G. This can be computed in time $\mathcal{O}\big(|E|\sqrt{|V|}\big)$. Hence finding the largest set of independent edges is easy, while finding the largest set of independent vertices is hard. In bipartite graphs, König's theorem says that the size of the smallest cover equals to the matching number. Hence, $\alpha(G) + \beta(G) = n$ in any bipartite graph.

A vertex coloring $c : V \to \{1, \dots, k\}$ of $G = (V, E)$ is such that $(x, y) \in E$ implies $c(x) \neq c(y)$. G is k-colorable if there exists a coloring c with codomain of size k. Equivalently, G is k-colorable if there exists a partition $V_1, \dots, V_k$ of V such that each subset V_i is independent in G. The **chromatic number** $\chi(G)$ is the smallest integer k such that G is k-colorable. The associated decision problem is NP-complete except for $k \le 2$. Here are some easy facts.

- Any bipartite graph is 2-colorable.
- For the clique, $\chi(K_n) = n$. This implies $\omega(G) \le \chi(G)$ for all G.
- For the cycle, $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$.

The four-color theorem says that $\chi(G) \le 4$ for all planar graphs G (a graph is planar when it can be drawn so that if any two edges intersect, they only do so in a vertex).

Clearly, $1 \le \chi(G) \le n$, and in general $\chi(G) \le 1 + \mathcal{O}\big(\sqrt{E}\big)$, as shown next.

Fact 1 *For all G,*

$$\chi(G) \le \frac{1}{2} + \sqrt{2|E| + \frac{1}{4}}$$

Proof. Let $k = \chi(G)$ and $V_1, \dots, V_k$ be the partition of V induced by the k-coloring. Due to the minimality of k, there is at least one edge in E for any pair (V_i, V_j) with $i \neq j$. Therefore, $|E| \ge \binom{k}{2}$. Solving for k gives the result. □

The chromatic number cannot be large when all vertices have small degrees.

Fact 2 *For all G, $\chi(G) \le \Delta(G) + 1$.*

The bound is tight when $G = K_n$ or $G = C_{2n+1}$.

Proof. Pick an arbitrary permutation $v_1, \ldots, v_n$ of the vertices in V. Consider the greedy algorithm colors each vertex v_i according to the permutation and using a new color only when $i = 1$ or when for each previously used color k, v_i has a neighbour in $v_1, \ldots, v_{i-1}$ which got colored with k. Since v_i has at most $\Delta(G)$ neighbours, then $\Delta(G) + 1$ colors are sufficient. □

Since each color uniquely corresponds to an independent set, a small independence number implies a large chromatic number.

Fact 3 *For all G, $\chi(G)\alpha(G) \geq |V|$.*

Proof. Let $k = \chi(G)$ and $V_1, \ldots, V_k$ the partition of V induced by a k-coloring of G. Since each V_i is independent in G,

$$|V| = \sum_{i=1}^{k} |V_i| \leq \sum_{i=1}^{k} \alpha(G) \leq \chi(G)\alpha(G)$$

concluding the proof. □

The next important result shows that a small average degree implies a large independence number.

Theorem 4 (Turán, 1941) *For all G, $\alpha(G)\big(d(G) + 1\big) \geq |V|$.*

We introduce the notation $C(v) = N(v) \cup \{v\}$ for the neighborhood of v that includes v.

Proof. Consider the following greedy algorithm to construct an independent set: pick the vertex of smallest degree and remove it from the graph together with its neighborhood. Iterate on the remaining graph until there are no more vertices to pick. Clearly, the set of picked vertices from an independent set of size bounded by $\alpha(G)$. Introduce the quantity

$$Q(G) = \sum_{v \in V} \frac{1}{1 + d(v)}$$

When the first vertex v_1 is picked by the algorithm, the above quantity goes down by

$$\sum_{u \in C(v_1)} \frac{1}{1 + d(u)} \leq \sum_{u \in C(v_1)} \frac{1}{1 + d(v_1)} = \frac{|C(v_1)|}{1 + d(v_1)} = 1$$

Let d_i be the degree function when the i-th vertex v_i is picked. Since $d_i(v) \leq d(v)$ for all $v \in V$,

$$\sum_{u \in C(v_i)} \frac{1}{1 + d(u)} \leq \sum_{u \in C(v_i)} \frac{1}{1 + d_i(u)} \leq \sum_{u \in C(v_i)} \frac{1}{1 + d_i(v_i)} = \frac{|C(v_i)|}{1 + d_i(v_i)} = 1$$

Since it takes at most $\alpha(G)$ steps until $Q(G)$ goes to zero, and we decrease $Q(G)$ be at most one in each step, $Q(G) \leq \alpha(G)$. Now we just observe that

$$\frac{1}{|V|} \sum_{v \in V} \frac{1}{1 + d(v)} \geq \frac{1}{1 + \frac{1}{|V|} \sum_{v \in V} d(v)} = \frac{1}{1 + d(G)}$$

because of Jensen's inequality applied to the convex function $f(x) = \frac{1}{1+x}$. □

Turán's theorem also shows that a small average degree implies a large clique number.

Corollary 5 *For all G, $\omega(G)\big(|V| - d(G)\big) \geq |V|$.*

PROOF. Let $n = |V|$ and $G' = (V, E')$ be the complement of G, where $e \in E'$ if and only if $e \notin E$. If $d(v)$ is the degree of v in G and $d(G')$ is the average degree in G', then

$$d(G') = \frac{1}{n}\sum_{v \in V}\big(n - 1 - d(v)\big) = n - 1 - d(G)$$

As an independent set in G' corresponds to a clique in G, we can apply Turán's theorem and obtain (for $n = |V|$),

$$\omega(G) = \alpha(G') \geq \frac{n}{d(G') + 1} = \frac{n}{n - d(G)}$$

concluding the proof. □

A subset $U \subseteq V$ is dominating in $G = (V, E)$ if every vertex in $V \setminus U$ has a neighbor in U. The **dominating number** $\gamma(G)$ is the size of the smallest dominating set in G.

it is fairly easy to show that the dominating number is always smaller than the independence number.

Fact 6 *For all G, $\gamma(G) \leq \alpha(G)$.*

PROOF. Let U be an independent set of maximum cardinality. Then U is a dominating set. Indeed, if U is not dominating then there is $x \in V$ without neighbors in U. But then we $U \cup \{x\}$ is an independent set larger than U. □

The next result shows that if all vertices have a large degree, then the dominating number must be small.

Theorem 7 (Arnautov, 1974; Payan, 1975) *For all G*

$$\gamma(G)\frac{1 + \delta(G)}{1 + \ln\big(1 + \delta(G)\big)} \leq |V|\ .$$

PROOF. Let $n = |V|$ and $\delta = \delta(G)$. We run a greedy algorithm choosing the vertices for the dominating set one by one, where in each step a vertex that covers the maximum number of yet uncovered vertices is picked, where an uncovered vertex does not lie in the union of the sets $C(v)$ of the vertices v chosen by the algorithm so far. Let $\mathcal{U}$ be the of currently uncovered vertices and let $r = |\mathcal{U}|$. Then the next vertex v chosen by the algorithm satisfies

$$\begin{aligned}
\max_{v \in V}\sum_{u \in \mathcal{U}}\mathbb{I}\{u \in C(v)\} &\geq \mathbb{E}\left[\sum_{u \in \mathcal{U}}\mathbb{I}\{u \in C(V)\}\right] \qquad (V \text{ chosen at random with uniform probability})\\
&= \sum_{u \in \mathcal{U}}\mathbb{P}\big(u \in C(V)\big)\\
&= \sum_{u \in \mathcal{U}}\frac{|C(u)|}{n}\\
&\geq \sum_{u \in \mathcal{U}}\frac{\delta + 1}{n} = \frac{r(\delta + 1)}{n}
\end{aligned}$$

Adding this v to the set of chosen vertices we observe that the number of uncovered vertices is now at most $r(1-(\delta+1)/n)$. It follows that in each iteration of the above procedure the number of uncovered vertices decreases by a factor of $1-(\delta+1)/n$. The number m of steps it takes so that $|\mathcal{U}| \leq \frac{n}{\delta+1}$ satisfies $n\big(1-\frac{\delta+1}{n}\big)^m \leq \frac{n}{\delta+1}$. Using $1-x \leq e^{-x}$ this implies

$$n \exp\left(-\frac{\delta+1}{n}m\right) \leq \frac{n}{\delta+1}$$

which solved for m gives $m \geq \frac{n}{\delta+1}\ln(\delta+1)$. Adding the set of $n/(\delta+1)$ yet uncovered vertices gives the desired result. □