Complementi di Algoritmi e Strutture Dati

First-price and second-price auctions

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This material is partially based on the book: Networks, Crowds, and Markets: Reasoning about a Highly Connected World, by David Easley and Jon Kleinberg. Cambridge University Press, 2010.

The underlying assumption we make when modeling auctions is that each bidder i has an intrinsic value $v_i \in [0,1]$ for the item being auctioned. They are willing to purchase the item for a price up to this value, but not for any higher price.

Ascending-bid auctions: These auctions (also called English auctions) are carried out interactively in real time. The seller gradually raises the price, bidders drop out until finally only one bidder remains, and that bidder wins the object at this final price. These correspond to second-price sealed-bid auctions, where bidders submit simultaneous sealed bids to the seller and the highest bidder wins the object paying the value of the second-highest bid.

Descending-bid auctions: This is also an interactive auction format (also called Dutch auction), in which the seller gradually lowers the price from some high initial value until the first moment when some bidder accepts and pays the current price. These correspond to first-price sealed-bid auctions, where bidders submit simultaneous sealed bids to the seller and the highest bidder wins the object paying the value of their bid.

A shading strategy $s : [0,1] \to [0,1]$ is a map from valuations to bids. As we assume that bidders are rational, $s(v) \le v$ holds for all $v \in [0,1]$. Hence $s(0) = 0$. We also assume that s is monotone: $v' > v$ implies $s(v') > s(v)$. In other words, we always increase a bid if the valuation increases.

Second-price auctions. If there are n bidders with valuations $v_1, \dots, v_n$ and using shading strategies $s_1, \dots, s_n$, the payoff function for bidder 1 in a second-price auction is

$$f_1(v_1, \dots, v_n, s_1, \dots, s_n) = \mathbb{I}\left\{ s_1(v_1) > \max_{i \neq 1} s_i(v_i) \right\} \left(v_1 - \max_{i \neq 1} s_i(v_i) \right)$$

and similarly for the other bidders.

We say that strategy s_1 is dominating for bidder 1 if

$$f_1(v_1, \dots, v_n, s_1, \dots, s_n) \ge f_1(v_1, \dots, v_n, s', \dots, s_n) \qquad \text{for all } v_1, \dots, v_n, s_1, \dots, s_n, s'$$

Teorema 1 *In a second-price auction, the strategy $s : v \mapsto v$ is dominating for any bidder.*

Dimostrazione. Consider bidder i with valuation v_i and bid b_i. Consider first $b_i > v_i$. If i is winning with $b_i = v_i$, then increasing the bid does not change the payoff. If i is losing with $b_i = v_i$, then then payoff remains zero unless the new bid goes above the highest bid $\max_{j \neq i} b_j > v_i$. In this case the payoff becomes negative. Hence i should not consider $b_i > v_i$. Now consider $b_i < v_i$. If i is losing with $b_i = v_i$, then decreasing the bid does not change the payoff. If i is winning with $b_i = v_i$, then then payoff remains $v_i - \max_{j \neq i} b_j > 0$ unless the new bid goes below the second-highest bid $\max_{j \neq i} b_j$. In this case the payoff becomes zero. Hence i should not consider $b_i < v_i$. This concludes the proof. □

First-price auctions. If there are two bidders with valuations v_1, v_2 and using shading strategies s_1, s_2, the payoff function for bidder 1 in a first-price auction is

$$g_1(v_1, v_2, s_1, s_2) = \mathbb{I}\{s_1(v_1) > s_2(v_2)\}\big(v_1 - s_1(v_1)\big)$$

and similarly for bidder 2.

Assuming the valuations v_1, v_2 are realizations of two random variables V_1, V_2, an equilibrium for the two bidders is a pair (s_1, s_2) of strategies such that

$$\begin{aligned} &\mathbb{E}\Big[g_1(v_1, V_2, s_1, s_2) - g_1(v_1, V_2, s', s_2)\Big] \ge 0 \\ &\mathbb{E}\Big[g_2(V_1, v_2, s_1, s_2) - g_2(V_1, v_2, s_1, s')\Big] \ge 0 \end{aligned} \qquad \text{for all } s', v_1, v_2$$

Teorema 2 *If the valuations V_1, V_2 for the two bidders are independently drawn from the uniform distribution over the $[0,1]$ interval and the two bidders use the same shading strategy s, then (s,s) with $s : v \mapsto v/2$ is an equilibrium for the bidders in a first-price auction.*

DIMOSTRAZIONE. Given the realized valuation v_1, the expected payoff for player 1 is

$$\begin{aligned} \mathbb{E}\big[g_1(v_1, V_2, s, s)\big] &= \mathbb{P}\big(s(v_1) > s(V_2)\big)\big(v_1 - s(v_1)\big) \\ &= \mathbb{P}\big(v_1 > V_2\big)\big(v_1 - s(v_1)\big) \\ &= v_1\big(v_1 - s(v_1)\big) \end{aligned}$$

where we used the monotonicity assumption: $s(v) > s(v')$ if and only if $v > v'$, and our assumption on the uniform distribution for V_1, V_2. The equilibrium condition for player 1 then states that

$$v_1\big(v_1 - s(v_1)\big) \ge v_1\big(v_1 - s'(v_1)\big)$$

Since bidder 2 is never going to bid more than $s(1)$, we can assume that s' satisfies the additional condition $s'(v) \in [0, s(1)]$ for all $v \in [0,1]$. Indeed, bidding higher than $s(1)$ would only reduce the payoff of bidder 1, without increasing the probability of winning. This implies that we can find $v \in [0,1]$ such that $s'(v_1) = s(v)$. Hence, the equilibrium condition becomes

$$v_1\big(v_1 - s(v_1)\big) \ge v_1\big(v_1 - s(v)\big)$$

Substituting $s(v) = v/2$ we get

$$\frac{v_1^2}{2} \ge v_1^2 - \frac{vv_1}{2}$$

Multiplying by 2 both sides and rearranging we obtain $v_1^2 + v^2 - 2vv_1 \ge 0$ which is always true. This concludes the proof. □