Lecture 2: The Graph Minor Theorem

25/05/2022

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The following result will be used below:

Theorem 0.1 (Robertson-Seymour)**.** *For every graph H there exists an algorithm that, for every G, decides if $H \preceq G$ in time $\mathcal{O}(|V(G)|^3)$.*

1 Minor-closed graph families

Recall that a *graph property* is a family of graphs closed under isomorphism. Many important properties are *minor-closed*, that is, closed under taking minors.

Definition 1.1. A graph family $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies $H \in \mathcal{F}$ for every $H \preceq G$.

Exercise 1. *Decide if the following graph properties are minor-closed: being acyclic, being planar, having maximum degree at most k, having diameter at most k.*

Theorem 1.2. *$\mathcal{F}$ is minor-closed if and only if $\mathcal{F} = \mathrm{Forb}(\mathcal{H})$ for some $\mathcal{H}$.*

Proof. Let $\overline{\mathcal{F}} = \{G : G \notin \mathcal{F}\}$ be the complement of $\mathcal{F}$.

If $\mathcal{F}$ is minor-closed then every $G \in \mathcal{F}$ satisfies $G \not\succeq H$ for all $H \in \overline{\mathcal{F}}$, while every $G \notin \mathcal{F}$ satisfies $G \succeq G \in \overline{\mathcal{F}}$. Hence $\mathcal{F} = \mathrm{Forb}(\overline{\mathcal{F}})$, which proves for $\mathcal{H} = \overline{\mathcal{F}}$ proves the claim.

Now let $\mathcal{F} = \mathrm{Forb}(\mathcal{H})$. If $\mathcal{F}$ is not minor-closed, then there is $G \in \mathcal{F}$ such that $G \succeq G'$ for some $G' \in \overline{\mathcal{F}}$, which thus satisfies $G' \succeq H$ for some $H \in \mathcal{H}$. But then by transitivity $G \succeq H$, which implies $G \notin \mathcal{F}$, a contradiction. □

Theorem 1.2 says that every minor-closed graph property has an obstruction set and viceversa.

2 The Robertson-Seymour theorem

The following result is among the deepest in graph theory:

Theorem 2.1 (The Robertson-Seymour graph minor theorem)**.** *In any infinite sequence of graphs $G_0, G_1, \ldots$ there are indices $i < j$ such that $G_i \preceq G_j$.*

Note that the claim Theorem 2.1 does not hold for the subgraph relation $\subseteq$ (why?). To appreciate Theorem 2.1 let us see two of its consequences.

Consequence #1.

Theorem 2.2. *$\mathcal{F}$ is minor-closed if and only if it has a finite obstruction set.*

Proof. The backward direction is trivial. For the forward direction, define:

$$\mathcal{H}_{\mathcal{F}} = \{H \mid H \in \overline{\mathcal{F}} \text{ and } \nexists H' \in \overline{\mathcal{F}} : H' \prec H\} \tag{1}$$

It is easy to see that $\mathcal{F} = \mathrm{Forb}(\mathcal{H}_{\mathcal{F}})$. Indeed, if $G \in \mathcal{F}$ then $G \not\succeq H$ for all $H \in \mathcal{H}_{\mathcal{F}}$ since $\mathcal{H}_{\mathcal{F}} \subseteq \overline{\mathcal{F}}$; if $G \notin \mathcal{F}$ then $G \succeq H$ for some $H \in \overline{\mathcal{F}}$, and by construction $\mathcal{H}_{\mathcal{F}}$ contains either H or a proper minor. Now let $H_1, H_2, \ldots,$ be any enumeration of $\mathcal{H}_{\mathcal{F}}$. By Theorem 2.1 $\mathcal{H}_{\mathcal{F}}$ must be finite, otherwise $H_i \prec H_j$ for some $i < j$, contradicting the definition of $\mathcal{H}_{\mathcal{F}}$. □

Consequence #2.

Theorem 2.3. *Every minor-closed graph property can be decided in time $O(|V(G)|^3)$.*

Proof. Let $\mathcal{F}$ be the property. By Theorem 2.2, $\mathcal{F}$ has a finite obstruction set $\mathcal{H}$. To decide whether any given G is in $\mathcal{F}$, list every $H \in \mathcal{H}$ and check whether $H \preceq G$ in time $\mathcal{O}(|V(G)|^3)$ using the algorithm of Theorem 0.1. Since $\mathcal{H}$ is fixed (i.e., not part of the input) then the running time is polynomial in the input size. □

This implies, for instance, the existence of a polynomial-time algorithm for planarity testing. Unfortunately, the constants hidden in the running time of the algorithm of Theorem 0.1 make the algorithm impractical.

3 Proof of the graph minor theorem for trees

The proof of Theorem 2.1 is nontrivial; it took two decades and several hundred pages. Here, we prove it for the special case of trees:

Theorem 3.1. *In any infinite sequence of trees there are two trees T, T' such that $T \preceq T'$.*

3.1 Colorings and monochromatic subsets

A k-coloring of a set A is a function $c : A \to [k]$. For any set X let $[X]^h$ be the set of all h-sized subsets of X. Thus, a k-coloring of $[X]^h$ assigns a color to every h-sized subset of X. Given a k-coloring of $[X]^h$, we say $Y \subseteq X$ is *monochromatic* if c is constant over $[Y]^h$ (for $k = 2$ think of a clique with vertex set X and a k-coloring c of the edges; a monochromatic subset is a sub-clique whose edges have all the same color).

Theorem 3.2. *Let $c, h \in \mathbb{N}$ and X an infinite set. If $[X]^h$ is coloured with c colors then X has an infinite monochromatic subset.*

Proof. We use induction on h. For $h = 1$ the claim is trivial. Let then $h > 1$ and assume the claim holds for $h - 1$. Let $X_0 = X$, choose any $x_0 \in X_0$, and consider $[X_0 \setminus \{x_0\}]^{h-1}$. We define a coloring of $[X_0 \setminus \{x_0\}]^{h-1}$ by letting $c(Z) = c(\{x_0\} \cup Z)$ for every $Z \in [X_0 \setminus \{x_0\}]^{h-1}$. By inductive hypothesis, there is an infinite $Y_0 \subseteq X_0 \setminus \{x_0\}$ such that every $Z \in [Y_0]^{h-1}$ has the same color; call it c_0. Clearly $c(\{x_0\} \cup Z) = c_0$ for all $Z \in [Y_0]^{h-1}$, too. Let $X_1 = Y_0$, choose any $x_1 \in X_1$, and repeat.

We obtain an infinite sequence of sets $X = X_0 \supseteq X_1 \supseteq \ldots$ and elements $(x_i)_{i \geq 0}$ with colors $(c_i)_{i \geq 0}$. As the colors are finite, there is an infinite set $Y = \{x_{i_j} : j \geq 0\}$ with the same color. By construction, $c(Z)$ is constant for all $Z \in [Y]^h$, hence Y is monochromatic. □

3.2 Well-quasi-orderings

A relation $\preceq$ over a set X is a *quasi-ordering* if it is:

- reflexive: $x \preceq x$ for all $x \in X$
- transitive: $x \preceq y$ and $y \preceq z$ implies $x \preceq z$, for all $x, y, z \in X$

(Note that the minor relation is a quasi-ordering). If neither $x \preceq y$ nor $y \preceq x$, then x and y are *incomparable*. A set of pairwise incomparable elements is an *antichain*. A sequence $(x_i)_{i \geq 0}$ is *decreasing* if $x_i \succ x_{i+1}$ for all $i \geq 0$, and is *nondecreasing* if $x_i \preceq x_{i+1}$ for all $i \geq 0$. Increasing and nonincreasing sequences are defined similarly. A sequence is *good* if it contains a good pair, that is, a pair of elements $x_i \preceq x_j$ with $i < j$; otherwise the sequence is *bad*. A quasi-ordering $\preceq$ on X is a *well-quasi-ordering* if every infinite sequence $x_0, x_1, \ldots$ over X is good.

Lemma 3.3. *X is well-quasi-ordered by $\preceq$ if and only if X contains neither an infinite antichain nor an infinite decreasing sequence.*

Proof. For the forward direction, if $\preceq$ is a well-quasi-ordering then by definition every infinite sequence contains a good pair and therefore cannot be an antichain or a decreasing sequence.

For the backward direction, let $(x_i)_{i \in \mathbb{N}}$ be any infinite sequence over X and consider the 3-coloring of $[\mathbb{N}]^2$ defined as follows, assuming without loss of generality that $i < j$:

$$c(\{i, j\}) = \begin{cases} 1 & x_i \preceq x_j \\ 2 & x_i \succ x_j \\ 3 & x_i, x_j \text{ incomparable} \end{cases} \tag{2}$$

By Theorem 3.2, there is an infinite $Y \subseteq \mathbb{N}$ such that all elements of $[Y]^2$ have the same colour. In other words there is an infinite subsequence of $(x_i)_{i \in \mathbb{N}}$ that is either increasing, or decreasing, or an antichain. But the last two possibilities are ruled out by hypothesis. Hence $(x_i)_{i \geq N}$ contains an infinite nondecreasing sequence. □

The proof above actually shows:

Corollary 3.4. *X is well-quasi-ordered by $\preceq$ if and only if every infinite sequence in X has an infinite nondecreasing subsequence.*

3.3 Well-quasi-orderings of finite subsets

For any set X we denote by $[X]^{<\omega}$ the set of all finite subsets of X. Every quasi-order $\preceq$ over X can be extended to $[X]^{<\omega}$: for every $A, B \in [X]^{<\omega}$ let $A \preceq B$ if and only if there is an injection $f : A \to B$ such that $a \preceq f(a)$ for all $a \in A$. It is easy to see that $\preceq$ is a quasi-order on $[X]^{<\omega}$.

Lemma 3.5. *If X is well-quasi-ordered by $\preceq$ then so is $[X]^{<\omega}$.*

Proof. Suppose X is well-quasi-ordered by $\preceq$ but $[X]^{<\omega}$ is not. Thus $[X]^{<\omega}$ contains an infinite sequence that is bad. We construct a bad infinite sequence $(A_i)_{i\geq 0}$ that shows that X is not well-quasi-ordered by $\preceq$, a contradiction. Let $A_0 \in [X]^{<\omega}$ be the smallest nonempty set such that there exists a bad infinite sequence in $[X]^{<\omega}$ starting with A_0. Now, for every $i = 0, 1, \ldots$, we choose $A_{i+1} \in [X]^{<\omega}$ of minimum cardinality such that there is a bad sequence in $[X]^{<\omega}$ starting with $A_0, \ldots, A_{i+1}$. The sequence $(A_i)_{i\geq 0}$ thus obtained is clearly a bad sequence.

For every $i \geq 0$ choose an arbitrary $a_i \in A_i$. By Corollary 3.4 the sequence $(a_i)_{i\geq 0}$ has an infinite nondecreasing subsequence $(a_{i_j})_{j\geq 0}$. For every $j \geq 0$ define $B_{i_j} = A_{i_j} \setminus \{a_{i_j}\}$, and consider the sequence:

$$S = A_0, \ldots, A_{i_0 - 1}, B_{i_0}, B_{i_1}, \ldots \tag{3}$$

This sequence S must be good, because if it was bad, then after choosing $A_0, \ldots, A_{i_0-1}$ we should have chosen B_{i_0} instead of A_{i_0}. Hence S contains a good pair. We claim that this implies $(A_i)_{i\geq 0}$ being good, a contradiction.

Choose any good pair in S. If the pair is in the form A_i, A_j then this implies directly that $(A_i)_{i\geq 0}$ is good. If the pair is in the form A_i, B_j then observe that $B_j \preceq A_j$, hence (by transitivity) $A_i \preceq A_j$, so A_i, A_j is again good. If the pair is in the form B_i, B_j then since $A_i = B_i \cup \{a_i\}$ and $A_j = B_j \cup \{a_j\}$, and since $a_i \preceq a_j$, then once again $A_i \preceq A_j$. Therefore in any case $(A_i)_{i\geq 0}$ is good, which is absurd since it was bad by construction. □

3.4 The proof

We can now prove the graph minor theorem for trees.

Theorem 3.6 (Kruskal, 1960)**.** *The set of finite trees is well-quasi-ordered by the minor relation.*

Proof. The proof actually gives a stronger claim: it holds for *rooted trees* under the following relation $\preceq$ which is a stronger version of the minor one. Given a tree T with root t, the *tree-order* $\leq$ over $V(T)$ is such that $x \leq y$ iff x lies on the path $T(r, y)$ between r and y. Given two trees T, T' with roots r, r', let $T \preceq T'$ iff there is an isomorphism φ from a subdivision of T to a subtree $T'' \subseteq T'$ that preserves the tree order, i.e., such that if $x \preceq y$ then $\varphi(x) \preceq \varphi(y)$. It is not hard to see that $\preceq$ is a quasi-ordering over the family of rooted trees.

Now suppose by contradiction that the claim was false. As in the proof of Lemma 3.5, construct a bad infinite sequence $(T_i)_{i\geq 0}$ of rooted trees by letting T_{i+1} be any smallest tree (i.e. with the fewest vertices) that extends $T_0, \ldots, T_i$. For every $i \geq 0$ let r_i be the root of T_i and let A_i be the set of rooted trees in $T_i \setminus r_i$ (the roots are the neighbors of r_i). We prove that $(A_i)_{i\geq 0}$ contains a pair A_i, A_j with $i < j$ such that for every $T \in A_i$ there is a distinct $T' \in A_j$ satisfying $T \preceq T'$. It is then easy to see that $T_i \preceq T_j$. Hence $(T_i)_{i\geq 0}$ contains a good pair, which is absurd.

Let $A = \cup_{i\geq 0} A_i$. We prove that A is well-quasi-ordered. By Lemma 3.3 this implies that $[A]^{<\omega}$ is well-quasi-ordered too, and therefore $(A_i)_{i\geq 0}$, which is an infinite subsequence over $[A]^{<\omega}$, contains a good pair. Let $(T^k)_{k\geq 0}$ be any infinite sequence in A. For every $k \geq 0$ choose $n(k)$ such that $T^k \in A_{n(k)}$, and let $k^* = \arg\min_{k\geq 0} n(k)$. Look at the sequence:

$$S = T_0, \ldots, T_{n(k^*)-1}, T^{k^*}, T^{k^*+1}, \ldots \tag{4}$$

Note that S is good: if it was bad, then in the construction of $(T_i)_{i \geq 0}$ we would have chosen T^k instead of $T_{n(k)}$, as $|V(T^k)| < |V(T_{n(k)})|$. The same arguments of the proof of Lemma 3.3 show that any good pair in S has the form T^{k_1}, T^{k_2}, and thus is a good pair in $(T^k)_{k \geq 0}$, as claimed. $\square$